

Markscheme

November 2020

Calculus

Higher level

Paper 3

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a valid **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown.
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the “must be seen” marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp **A0** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, eg **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (eg substitution into a formula) and **A1** for using the **correct** values.
- Where the markscheme specifies (**M2**), **N3**, etc., do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final **A1**. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct **FT** working shown, award **FT** marks as appropriate but do not award the final **A1** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	Award the final A1 (ignore the further working)
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final A1
3.	$\log a - \log b$	$\log(a - b)$	Do not award the final A1

3 N marks

Award **N** marks for **correct** answers where there is **no** working.

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (eg $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses [**1 mark**].

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (eg $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks (d)

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation **DM** should be used and a brief **note** written next to the mark explaining this decision.

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives

$$f'(x) = (2\cos(5x - 3))5 (=10\cos(5x - 3)) \quad \mathbf{A1}$$

Award **A1** for $(2\cos(5x - 3))5$, even if $10\cos(5x - 3)$ is not seen.

10 Accuracy of Answers

Candidates should **NO LONGER** be penalized for an accuracy error (**AP**).

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

1. attempt to use l'Hôpital's rule **M1**

$$= \lim_{x \rightarrow 1} \frac{-2x \sin(x^2 - 1)}{e^{x-1} - 1} \quad \text{A1A1}$$

Note: Award **A1** for the numerator and **A1** for the denominator.

substitution of 1 into their expression **(M1)**

$$= \frac{0}{0} \text{ hence use l'Hôpital's rule again}$$

Note: If the first use of l'Hôpital's rule results in an expression which is not in indeterminate form, do not award any further marks.

attempt to use product rule in numerator **M1**

$$= \lim_{x \rightarrow 1} \frac{-4x^2 \cos(x^2 - 1) - 2 \sin(x^2 - 1)}{e^{x-1}} \quad \text{A1}$$

$$= -4 \quad \text{A1}$$

[7 marks]

2. (a) **METHOD 1**

$$\frac{a_{n+1}}{a_n} = \frac{(5-3x)^{n+1}}{n+1} \cdot \frac{n}{(5-3x)^n} \quad \text{M1}$$

$$= \frac{n}{n+1}(5-3x) \quad \text{A1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |5-3x| \quad \text{(M1)}$$

Note: Award **(M1)** for finding the limit of their expression.

for convergence, $|5-3x| < 1$ **R1**

Note: Award **R1** for $|\text{their limit}| < 1$.

$$-1 < 5-3x < 1 \text{ or } -\frac{1}{3} < \frac{5}{3} - x < \frac{1}{3} \quad \text{A1}$$

Note: If a candidate writes a correct inequality, without errors in working, but did not explicitly write the absolute value of their limit, award **ROA1**.

$$\frac{4}{3} < x < 2 \quad \text{AG}$$

METHOD 2

$$\lim_{n \rightarrow \infty} \left| \frac{(5-3x)^n}{n} \right|^{\frac{1}{n}} \quad \text{M1}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{5-3x}{n^{\frac{1}{n}}} \right| \quad \text{A1}$$

$$= |5-3x| \quad \text{A1}$$

for convergence, $|5-3x| < 1$ **R1**

Note: Award **R1** for $|\text{their limit}| < 1$.

$$-1 < 5-3x < 1 \text{ OR } -\frac{1}{3} < \frac{5}{3} - x < \frac{1}{3} \quad \text{A1}$$

$$\frac{4}{3} < x < 2 \quad \text{AG}$$

Note: If the working for parts (a) and (b) is interchanged, award marks as per the markscheme, but award **A0** for the final mark in each case.

[5 marks]

(b) consider one of the endpoints **M1**

at $x = \frac{4}{3}$, series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic or p -series) **A1**

at $x = 2$, series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ **A1**

Note: If candidates omit sigma, award **A0A1**, but subsequent marks can still be awarded.

EITHER

$\left(\frac{1}{n+1} < \frac{1}{n} \Rightarrow \right)$ decreasing and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ hence series converges at $x = 2$ **R1**

OR

which converges by the alternating series test **R1**

THEN

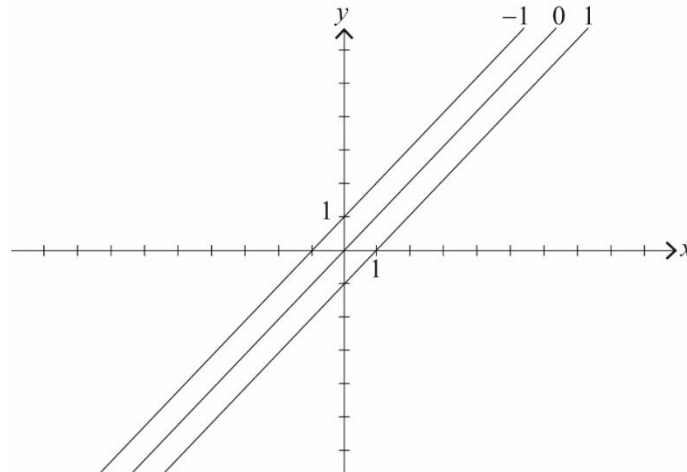
interval is $\frac{4}{3} < x \leq 2$ **A1**

[5 marks]

Total [10 marks]

3. (a) (i) attempt to find equation of isoclines by setting $x - y = -1, 0, 1$

M1



3 parallel lines with positive gradient

A1

y-intercept = $-c$ for $\frac{dy}{dx} = c$

A1

Note: To award **A1**, each y-intercept should be clear, but condone a missing label (eg.(0, 0)).

If candidates represent the lines using slope fields, but omit the lines, award maximum of **M1A0A1**.

(ii) at point (1, 1), $\frac{dy}{dx} = 0$

A1

EITHER

to the left of (1, 1), the gradient is negative

R1

to the right of (1, 1), the gradient is positive

R1

Note: Accept any correct reasoning using gradient, isoclines or slope field.

If a candidate uses left/right or $x < 1 / x > 1$ without explicitly referring to the point (1, 1) or a correct region on the diagram, award **R0R1**.

OR

$$\frac{d^2y}{dx^2} = 1 - \frac{dy}{dx}$$

A1

$$\frac{d^2y}{dx^2} = 1 (> 0)$$

A1

Note: accept correct reasoning that $\frac{dy}{dx}$ is increasing as x increases.

THEN

hence (1, 1) is a local minimum

AG

[6 marks]

- (b) integrating factor = $e^{\int dx}$ (M1)
- = e^x (A1)
- $\frac{dy}{dx}e^x + ye^x = xe^x$ (M1)
- $ye^x = \int xe^x dx$ A1
- = $xe^x - \int e^x dx$ (M1)
- = $xe^x - e^x (+c)$ A1

Note: Award **A1** for the correct RHS.

substituting (1, 1) gives

$e = e - e + c$ M1

$c = e$

$y = x - 1 + e^{1-x}$ A1

[8 marks]

- (c) (i) **METHOD 1**
- EITHER**
- attempt to solve for the intersection $x - 1 + e^{1-x} = x - 1$ (M1)
- OR**
- attempt to find the difference $x - 1 + e^{1-x} - (x - 1)$ (M1)
- THEN**
- $e^{1-x} > 0$ for all x R1

Note: Accept $e^{1-x} \neq 0$ or equivalent reasoning.

therefore the curve does not intersect the isocline AG

METHOD 2

$y = x - 1$ is an (oblique) asymptote to the curve

R1

Note: Do not accept “the curve is parallel to $y = x - 1$ ”.

$y = x - 1$ is the isocline for $\frac{dy}{dx} = 1$

R1

therefore the curve does not intersect the isocline

AG

METHOD 3

The initial point is above $y = x - 1$, so $\frac{dy}{dx} < 1$

R1

$\Rightarrow x - y < 1$

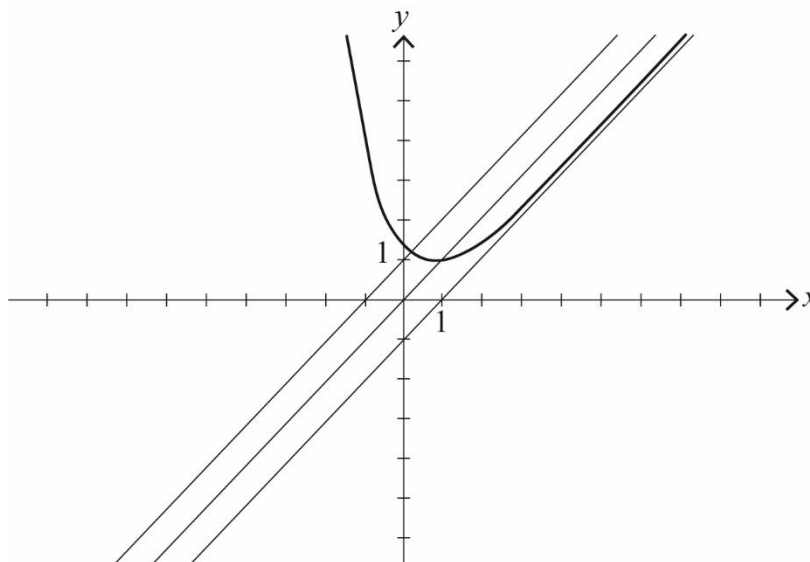
$\Rightarrow y > x - 1$

R1

therefore the curve does not intersect the isocline

AG

(ii)



concave up curve with minimum at approximately (1, 1)

A1

asymptote of curve is isocline $y = x - 1$

A1

Note: Only award **FT** from (b) if the above conditions are satisfied.

[4 marks]
Total [18 marks]

4. (a) (i) substitution of x^2 in $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ **(M1)**

$$x^2 - \frac{x^4}{2} + \frac{x^6}{3}$$
A1

(ii) $\frac{d}{dx}(\ln(1+x^2)) = \frac{2x}{1+x^2}$ **(M1)**

Note: Award **(M1)** if this is seen in part (a)(i).

attempt to differentiate their answer in part (a) **(M1)**

$$\frac{2x}{1+x^2} = 2x - \frac{4x^3}{2} + \frac{6x^5}{3}$$
M1

Note: Award **M1** for equating their derivatives.

$$\frac{x}{1+x^2} = x - x^3 + x^5$$
A1

[6 marks]

(b) $f(0.4) \approx 0.149$ **A1**

Note: Accept an answer that rounds correct to 2 s.f. or better.

[1 mark]

(c) (i) attempt to find the maximum of $|f^{(7)}(c)|$ for $c \in [0, 0.4]$ **(M1)**

maximum of $|f^{(7)}(c)|$ occurs at $c = 0.199$ **(A1)**

$|f^{(7)}(c)| < 1232.97\dots$ (for all $c \in]0, 0.4[$) **(A1)**

use of $x = 0.4$ **(M1)**

substitution of $n = 6$ **and** $a = 0$ **and** their value of x **and** their value of $f^{(7)}(c)$
into Lagrange error term **(M1)**

Note: Award **(M1)** for substitution of $n = 3$ **and** $a = 0$ **and** their value of x **and** their value of $f^{(4)}(c)$ into Lagrange error term.

$$|R_6(0.4)| < \frac{1232.97(0.4)^7}{7!}$$

upper bound = 0.000401 **A1**

Note: Accept an answer that rounds correct to 1 s.f or better.

(ii) $f^{(7)}(c) < 0$ (for all $c \in]0, 0.4[$) **R1**

Note: Accept $R_6(c) < 0$ or "the error term is negative".

the answer in (b) is an overestimate **A1**

Note: The **A1** is dependent on the **R1**.

[8 marks]
Total [15 marks]